Practice Final Exam
Math 110, Spring 2013
UC Berkeley

April 21, 2013

Exercise 1 (Tfheorem question starter)

(3 points) State and prove the Cauchy-Schwartz inequality for inner product spaces.

Exercise 2 (second Theorem question starter)

(3 points) Let V be a finite dimensional vector space and let T be a linear map from V to another vector space W (not necessarily finite dimensional). Show that

$$\dim V = \dim \mathbf{null} T + \dim \mathbf{range} T.$$

Exercise 3 (Peyam's multiple choice special)

(2 points) No answer, no points; a wrong answer removes one point. Circle either T (when you think the statement is true) or F (when you think the statement is false or makes no sense).

T/F If U_1, U_2, and U_3 are subspaces of V such that $V = U_1 + U_2 + U_3$ and $U_1 \cap U_2 = U_2 \cap U_3 = U_1 \cap U_3 = \{0\}$, then $V = U_1 \oplus U_2 \oplus U_3$.

T/F The set of all eigenvectors of a linear map T is a vector space.

T/F The minimal polynomial of an operator T divides the characteristic polynomial of T.

T/F If $T \in \mathcal{L}(V)$ is a self-adjoint operator, then there exists a basis of V consisting of eigenvectors of T.

T/F Every operator on an odd-dimensional vector space has a real eigenvalue.

T/F Every vector space is the direct sum of its generalized eigenspaces.

Exercise 4 (Mike's worksheet special)

(3 points) Consider two nilpotent operators S and T on a complex finite dimensional vector space. Prove that if $TS = ST$ then $S + T$ is nilpotent.

Exercise 5 (Daniel's worksheet special)

(3 points) Let V be a vector space and $U \subset V$ a subspace. We call a projection on U an operator $P : V \to V$ such that $P^2 = P$ and $\mathbf{range}P$=U. Show that P is self-adjoint if and only if $\mathbf{null}P$ is orthogonal to $\mathbf{range}P$ (in other words, P is an orthogonal projection).

Exercise 6 (Peyam's savory special)

(3 points) Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and consider the linear transformation $T : \mathbf{Mat}(2, 2, \mathbb{R}) \longrightarrow \mathbf{Mat}(2, 2, \mathbb{R})$ given by $T(B) = AB$. Find the matrix of T with respect to the standard basis

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

of the vector space $\mathbf{Mat}(2, 2, \mathbb{R})$ of real 2×2 matrices.

Exercise 7 (Mike's spicy special)

(3 points) Let V be a vector space of dimension n. Prove that an operator $D \in \mathcal{L}(V)$ is diagonal if and only if $D = \sum_{i=1}^{n} \lambda_i K_i$ where $\lambda_1, \ldots, \lambda_n$ are scalars and $K_i \in \mathcal{L}(V)$ for $i = 1, \ldots, n$ are operators such that $K_i^2 = K_i$ and such that $K_i K_j = 0$ if $i \neq j$.